Simplification of the direct interaction equations for turbulent shear flow

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Finally, we consider the *modified direct correlation function* $\hat{c}(\boldsymbol{x}_1, \boldsymbol{x}_2, N, \Omega, \gamma)$, defined in terms of $\hat{u}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, N, \Omega, \gamma)$ by

$$\int \mathrm{d}\boldsymbol{x}\, \hat{u}_2(\boldsymbol{x}, \boldsymbol{x}_1)\hat{c}(\boldsymbol{x}, \boldsymbol{x}_2) \equiv -\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{25}$$

Its space average $\bar{c}(\boldsymbol{r}, \rho, \gamma)$ can be defined like (2). We define the *weighted direct correlation function*

$$\bar{c}^{\mathrm{W}}(\boldsymbol{s}, \rho) \equiv \lim_{\gamma \to 0} \gamma^{-\nu} \bar{c}\left(\frac{\boldsymbol{s}}{\gamma}, \rho, \gamma\right). \tag{26}$$

Then, for one-phase ordered states, one can deduce from (20), (22) and (25) that

$$\bar{c}^{\mathrm{W}}(\boldsymbol{s}, \rho) = -\beta K(\boldsymbol{s}) - \beta\delta(\boldsymbol{s}) \frac{1}{\Gamma_\rho} \int_{\Gamma_\rho} \mathrm{d}\boldsymbol{y}\, a_2{}^0\{n_\rho(\boldsymbol{y})\}. \tag{27}$$

For one-phase fluid states one just puts $n_\rho = \rho$. Hence, for all one-phase states, we have our most important result

$$\bar{c}^{\mathrm{W}}(\boldsymbol{s}, \rho) = -\beta K(\boldsymbol{s}) \qquad \text{for } \boldsymbol{s} \neq 0 \tag{28}$$

which has been obtained in different forms by Lebowitz and Percus (1963) and Lebowitz *et al.* (1965).

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Simplification of the direct interaction equations for turbulent shear flow

Abstract. Simplifying assumptions can be made which reduce the full statistical equations for turbulent shear flow to differential equations which are amenable to computation.

Kraichnan (1966—this gives references to his earlier papers) first published his Direct Interaction (DI) method for closing the Navier–Stokes equations in 1958, and Edwards (1964) introduced the rather similar Fokker–Planck method six years later.

At the moment these methods cannot be justified formally, but increasingly it is being accepted that both their nature and the results which they give are reasonable.

So far, work has been more or less confined to homogeneous isotropic turbulence, since this rather artificial situation simplifies the algebra very greatly but still preserves the essence of the problem. In principle, there is no problem in extending DI to real situations such as turbulent shear flow, and Kraichnan (1964) has written out the governing equations. In practice, the dimensionality of the problem increases so much when the restrictions of homogeneity and isotropy are dropped that the equations have so far resisted analysis: they are probably too big for existing computers. The purpose of this letter is to introduce additional approximations which, it is hoped, will simplify the DI equations for real flows sufficiently to allow solutions to be obtained.

As an example I shall consider fully developed flow in a parallel-sided channel, but the methods should be equally applicable to any other problem. The correlation function $\tilde{q}_{ij}(\boldsymbol{x}, \boldsymbol{x}') = \langle \tilde{u}_i(\boldsymbol{x})\, \tilde{u}_j(\boldsymbol{x}') \rangle$ is then of the form

$$\tilde{q}_{ij}(\boldsymbol{x}, \boldsymbol{x}') \equiv \tilde{q}_{ij}(\boldsymbol{x} - \boldsymbol{x}', Y).$$

Here the tilde denotes a fluctuation velocity and $Y = \frac{1}{2}(x_2 + x_2')$, the 2 direction being normal to the channel walls. The first step is to Fourier transform with respect to the difference coordinate $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$, and the calculations are made on the 'semi-transform' $q_{ij}(\boldsymbol{k}, Y)$. This can be divided according to

$$q_{ij}(\boldsymbol{k}, Y) = q_{ij}{}^{\mathrm{S}}(\boldsymbol{k}, Y) + q_{ij}{}^{\mathrm{A}}(\boldsymbol{k}, Y) \tag{1}$$

where the S and A components are respectively symmetric and antisymmetric in $\boldsymbol{k}$ space. The q^{S} component is responsible for the energy of the turbulence, while q^{A} is responsible for the Reynolds stress. The first assumption is that q^{S} is isotropic

$$q_{ij}{}^{\mathrm{S}}(\boldsymbol{k}, Y) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) f(k, Y) \tag{2}$$

while q^{A} is obtained by applying a uniform strain to q^{S} (see Crow 1967). There must of course be other components present, with other angular symmetries in $\boldsymbol{k}$ space: I am guessing that they have relatively little effect on the turbulent intensity and on the Reynolds stress.

The next assumption is that the shape of the spectrum function $f(k, Y)$ is the same at all points of the flow, and that only the intensity and the width vary from one point to another. Some assumption of this form seems essential if the calculation is ever to be made simple enough for engineering-type applications. The spectral width specifies $K(Y)$, the reciprocal of the local length scale, while the intensity is conveniently given in terms of the local dissipation $\mathscr{E}(Y)$. We therefore assume

$$f(k, Y) = \frac{C}{4\pi} \{\mathscr{E}(Y)\}^{2/3} \{K(Y)\}^{-11/3} F\left[\frac{k}{K(Y)}\right] \tag{3}$$

where C is the Kolmogorov constant while $F(z)$ is a universal function which tends to $z^{-11/3}$ for large z, ensuring that $f(k, Y)$ goes over to the inertial range form $(C/4\pi)\mathscr{E}^{2/3}k^{-11/3}$ for large k. Similar assumptions are made about q^{A} and about Kraichnan's infinitesimal response function. The intensity of q^{A} is the Reynolds stress $\tau(Y)$ and, in the limit of zero viscosity, this is specified by Reynolds' equation. This makes an assumption of the form (3) particularly convenient.

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Finally, I assume that the variation of f and other spectral functions with k is much stronger than the variation with Y. This assumption is used to simplify the operator $P_{ijm}(\partial/\partial x_{\rm a})$ which appears in the configuration space version of the DI equations (see Kraichnan 1964). We put

$$\frac{\partial}{\partial x_{\rm a}} = \frac{\partial}{\partial r_{\rm a}} + \tfrac{1}{2}\delta_{\rm a2}\frac{\partial}{\partial Y}$$

where $\boldsymbol{r}$ is the separation $\boldsymbol{x}-\boldsymbol{x}'$, and treat the second term as a small perturbation. After Fourier-transforming with respect to $\boldsymbol{r}$ only, P_{ijm} becomes

$$P_{ijm}(\boldsymbol{k}) + R_{ijm}(\boldsymbol{k})\frac{\mathrm{d}}{\mathrm{d}Y} \tag{4}$$

further terms being ignored. The form of $P_{ijm}(\boldsymbol{k})$ is given by Kraichnan (1959), while $R_{ijm}(\boldsymbol{k})$ is a new function.

With all these assumptions the DI energy equation (the equation for the trace of q when $\boldsymbol{r} = 0$) simplifies enormously, and becomes

$$\mathscr{E} - A_1\frac{\mathrm{d}}{\mathrm{d}Y}\left(\frac{1}{K^2}\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}Y}\right) - A_2\frac{\mathrm{d}}{\mathrm{d}Y}\left(\frac{\mathscr{E}}{K^3}\frac{\mathrm{d}K}{\mathrm{d}Y}\right) = \tau U' \tag{5}$$

$U(Y)$ being the mean velocity. $\tau U'$ gives the production of turbulent energy, while $\mathscr{E}$ represents the dissipation of this energy. The two gradient-diffusion terms represent the spatial transport of this energy. They vanish in a logarithmic region in which both $\mathscr{E}$ and K are proportional to $1/Y$. This is particularly satisfactory, since experiments indicate that there is no net transport of turbulent energy in such a region. The constants A_1 and A_2 are determined once the forms of functions such as $F(z)$ are specified, and in this respect the present work is quite different from all previous theories of shear flow turbulence.

Equation (5) determines the turbulent intensity, and there is a similar relation between τ and U' for the off-diagonal components of q which actually determines U' (τ being known). A third equation is needed to determine K, and the form which this should take is being studied. Consideration is also being given to the point, made by Bradshaw *et al.* (1967), that the spatial transport of turbulent energy is partly a gradient process and is not wholly of the gradient-diffusion type. It seems that (5) is only valid for the small eddies, and that gradient terms should indeed be added to this equation to allow for the 'constriction' of the big eddies by the finite size of the system. In due course I hope to publish a paper dealing with these two points. In this paper I shall also try to justify statements which, because space is limited, have had to be made without proof in this letter.

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